VI.—*On the Degree of the Surface reciprocal to a given one.* *By the* Rev. GEORGE SALMON, *Fellow of Trinity College, Dublin.*

Read November 30, 1855.

I.—*General Theory.*

IT is my intention in the present memoir to lay before the Academy the extension and completion of a theory of reciprocal surfaces, the first outlines of which I published some years ago in the "Cambridge and Dublin Mathematical Journal" (vol. ii. p. 65, and vol. iv. p. 187). I there showed how to calculate the degree of the reciprocal of a surface having an ordinary double line; it remains now to show what the degree will be when the surface has likewise a cuspidal line (that is to say, a double line, the two tangent planes at every point of which coincide). I purpose next to examine the nature and number of those singular tangent planes to a surface which give rise to multiple points and lines on the reciprocal surface, and thus to show how it is that the degree of the reciprocal of that reciprocal coincides with the degree of the original surface. We shall thus obtain results analogous to the well-known theorems of M. PLUCKER for the case of curves. Lastly, I purpose to apply this theory to the case of developable surfaces, and to show how it is that the degree of the reciprocal of a developable reduces to nothing. I may mention that the substance of the present paper was prepared for publication in the year 1849, though various causes have prevented its being published until now.

I use the following notation for the following quantities, which will come under consideration in the discussion of the problems which it is proposed to investigate, and which may be regarded as the ordinary singularities of surfaces :—

Let n denote the order of the surface, that is to say, the number of points in which an arbitrary line meets it.

n', the class of the surface, that is to say, the number of tangent planes which can be drawn through an arbitrary right line.

a, the order of the tangent cone, drawn from any point to the surface.

δ, the number of its double edges.

κ, the number of its cuspidal edges.

b, the order of any double curve which may exist on the surface.

k, the number of its apparent double points; that is to say, the number of lines which can be drawn from an arbitrary point, twice intersecting the double curve.

t, the number of triple points on the double curve, which are also triple points on the surface.

c, the order of any cuspidal curve which may exist on the surface.

h, the number of its apparent double points.

β, the number of intersections of the double and cuspidal curve which are stationary points on the latter.

γ, the number of intersections which are stationary points on the former.

i, the number of intersections which are singular points on neither.

ρ, the number of points where the double curve b is met by the curve of contact a.

σ, the number of points where the cuspidal curve c is met by a.

Let the same letters accented denote the corresponding singularities of the reciprocal surface.

Having made this preliminary enumeration, I give in the first place the theory of the reciprocal of a surface of the n^{th} degree, having no multiple line. To know the nature of a section of the reciprocal surface, it is only necessary to know the nature of a tangent cone to the original surface. Having the degree of this cone, and the number of its double and cuspidal edges, we shall know at once by M. PLUCKER's formulæ, the characteristics of its reciprocal, namely, the section of the reciprocal surface.

I investigate the equation of the tangent cone by the method which M. JOACHIMSTAL has given for plane curves ("Crelle," vol. XXXIV. p. 24). Let the quadriplanar co-ordinates of two points be $xyzw$, $x'y'z'w'$, then—

$$\lambda x+\mu x', \qquad \lambda y+\mu y', \qquad \lambda z+\mu z', \qquad \lambda w+\mu w',$$

are the co-ordinates of any point on the line joining them. If these values be substituted for the current co-ordinates in the equation of any surface, the resulting equation solved for $\lambda:\mu$ gives the co-ordinates of the points where this line meets the surface. If then we form the condition $\phi U=0$, that this equation in $\lambda:\mu$ should have equal roots, and in it consider $xyzw$ as variable, it will represent the locus of all points, such that the line joining them to $x'y'z'w'$ touches the given surface; or, in other words, it will be the equation of the tangent cone whose vertex is $x'y'z'w'$.

If the equation of the surface be $U=0$, the result of this substitution will be

$$[U]=\lambda^n U+\lambda^{n-1}\mu\,\Delta U+\frac{\lambda^{n-2}\mu^2}{1.2}\Delta^2 U+\&c.=0\ ;$$

where Δ denotes the symbol

$$x'\frac{d}{dx}+y'\frac{d}{dy}+z'\frac{d}{dz}+w'\frac{d}{dw}.$$

The result of elimination then between $\frac{d[U]}{d\lambda}$ and $\frac{d[U]}{d\mu}$ is the equation of the tangent cone. It will obviously be of the $n\,(n-1)^{\text{th}}$ degree.

Cuspidal edges on the tangent cone arise when any edge of the cone meets the surface in three coincident points. If $xyzw$ be the co-ordinates of the point of contact, the equation $[U]=0$ must in this case be divisible by μ^3. The point must, therefore, be one of the intersections of the three surfaces $U=0$, $\Delta U=0$, $\Delta^2 U=0$; and since these are of the degrees n, $n-1$, $n-2$, the number of such points is $n\,(n-1)\,(n-2)$.

Double edges on the cone arise when any side touches the surface in two distinct points. The equation $[U]$ will in this case have two values of $\mu=0$, and two of the remaining values of μ equal. The co-ordinates then of any point of contact of a double tangent must satisfy the equations $U=0$, $\Delta U=0$, and $\psi U=0$, where ψU is the condition that the equation,

$$\frac{1}{1.2}\lambda^{n-2}\Delta^2 U+\frac{1}{1.2.3}\lambda^{n-3}\mu\,\Delta^3 U+\&c.=0,$$

should have equal roots. ψU is evidently of the degree $(n-2)\,(n-3)$ in $xyzw$.

The number, therefore, of points of contact of double tangents is $n(n-1)(n-2)(n-3)$; and the number of double tangents is of course half this. We have proved, then, that the degree of the tangent cone is $n(n-1)$; that it has $n(n-1)(n-2)$ cuspidal edges, and $\frac{n(n-1)(n-2)(n-3)}{2}$ double edges. The degree of its reciprocal, then, is, by M. PLUCKER's formula,

$$n(n-1)\{n(n-1)-1\}-3n(n-1)(n-2)-n(n-1)(n-2)(n-3) \\ =n(n-1)^2.$$

And since the degree of the reciprocal surface is the same as the degree of a plane section of it, we have in general $n'=n(n-1)^2$.

Let us now proceed to the case where the given surface has multiple lines. It appears by the same reasoning as for plane curves, that every line joining the point $x'y'z'w'$ to any point of a multiple line must be regarded as, in one sense, a tangent line to the surface: and that the cone determined by the equation $\phi U=0$ includes doubly the cone standing on the double curve b, and trebly the cone standing on the cuspidal curve c. If then a be the degree of the tangent cone proper, we have

$$n(n-1)=a+2b+3c.$$

To find the multiple edges of the tangent cone, we have, as before, to examine the points where the line of contact meets the two surfaces $\Delta^2 U$, and ψU. But the line of contact now consists of the complex line $a+2b+3c$, and the points where b and c meet $\Delta^2 U$ and ψU are plainly irrelevant to the question. Neither shall we have cuspidal or double edges answering to all the points where a meets these surfaces: since, if for example, any side of the cone a be also a side of the cone b, this must be considered as a double edge of the complex cone, although not a double line either on a or b. And any line passing through an intersection of the curves a and c must be considered as a cuspidal edge of the complex cone, although not so on either of the cones considered separately.

The following formulæ will be found to contain an analysis of the intersections of each of the curves a, b, c with the surface $\Delta^2 U$. The signification of the letters employed has been already explained :—

$$\begin{aligned} a\,(n-2) &= \kappa + \ \ \rho + 2\sigma, \\ b\,(n-2) &= \rho + 2\beta + 3\gamma + 3t, \\ c\,(n-2) &= 2\sigma + 4\beta + \ \ \gamma. \end{aligned} \tag{A}$$

I am not prepared to give a satisfactory explanation *à priori* of the numerical coefficients in these formulæ. I have obtained them by induction from an examination of a variety of particular surfaces. In particular, the surface which is the reciprocal of a surface of the third degree, and whose singularities can, without difficulty, be determined, furnished all the formulæ except the coefficient of γ, there being no points γ on this surface.

We derive, then, from these equations (A),

$$\kappa = (a-b-c)\ (n-2) + 6\beta + 4\gamma + 3t.$$

But since, if the surface had no multiple lines, the number of cuspidal edges in the tangent cone would be $(a+2b+3c)\ (n-2)$, the diminution in these caused by the double lines is

$$(3b+4c)\ (n-2) - 6\beta - 4\gamma - 3t.$$

Next, to find the number of double edges in the cone a. I use the symbol $[ab]$ to denote the number of apparent intersections of a and b, that is to say, the number of points where these two lines, seen from any point of space, appear to intersect, though they do not actually do so. The following formulæ, then, contain an analysis of the intersections of a, b, c with ψU:

$$\begin{aligned} a\,(n-2)\ (n-3) &= 2\delta + 3\,[ac] + 2\,[ab], \\ b\,(n-2)\ (n-3) &= 4k + \ \ [ab] + 3\,[bc], \\ c\,(n-2)\ (n-3) &= 6h + \ \ [ac] + 2\,[bc]. \end{aligned} \tag{B}$$

Hence,

$$2\delta = (a-2b-3c)\ (n-2)\ (n-3) + 8k + 18h + 12\,[bc].$$

But the number of apparent intersections of two curves is at once deduced from the number of their actual intersections. For if cones be described having a common vertex, and standing on the two curves, the common sides of these cones must answer either to apparent or actual intersections. Hence,

$$\begin{aligned}[ab] &= ab - 2\rho \\ [ac] &= ac - 3\sigma \\ [bc] &= bc - 3\beta - 2\gamma - i.^* \end{aligned} \qquad (C)$$

Substituting this value for $[bc]$, we have

$$2\delta = (a - 2b - 3c)\ (n-2)\ (n-3) + 8k + 18h + 12bc - 36\beta - 24\gamma - 12i,$$

and since, if the curve had no multiple lines, twice the number of double edges in the tangent cone would be $(a + 2b + 3c)\ (n-2)\ (n-3)$; the diminution in 2δ, caused by the double lines, is

$$(4b + 6c)\ (n-2)\ (n-3) - 8k - 18h - 12bc + 36\beta + 24\gamma + 12i.$$

By the help of the equations (C), the equations (B) may be written in the following form, which is sometimes more convenient:

$$\begin{aligned} a\,(n-2)\ (n-3) &= 2\delta + 2ab + 3ac - 4\rho - 9\sigma, \\ b\,(n-2)\ (n-3) &= 4k + ab + 3bc - 9\beta - 6\gamma - 3i - 2\rho, \\ c\,(n-2)\ (n-3) &= 6h + ac + 2bc - 6\beta - 4\gamma - 2i - 3\sigma. \end{aligned} \qquad (D)$$

It is easy now to find the effect of the lines b and c on the degree of the reciprocal surface. If the degree of a cone diminish from m to $m - l$, that of its reciprocal will diminish from $m\,(m-1)$ to $(m-l)\ (m-l-1)$; that is to say, will diminish by $l\,(2m - l - 1)$. In the present case $m = n^2 - n$, and $l = 2b + 3c$. The diminution then in the degree of the reciprocal, arising from the diminution in the degree of the tangent cone, is

$$(2b + 3c)\ (2n^2 - 2n - 2b - 3c - 1).$$

We must subtract from this three times the diminution in the number of cusps, together with twice the diminution in the number of double edges, and we find, for the total diminution of the degree of the reciprocal surface,

* It is proper to observe, that if the surface have a nodal curve (b), but no cuspidal curve, there will still be a determinate number i of cuspidal points on the nodal curve, and the equation given will receive the modification $[ab] = ab - 2\rho - i$. As, however, the quantity $[ab]$ is eliminated from the equations in finding the diminution in the degree of the reciprocal surface, the ultimate result is not affected.

$$(2b+3c)(2n^2-2n-2b-3c-1)-3(3b+4c)(n-2)+18\beta+12\gamma+9t$$
$$-(4b+6c)(n-2)(n-3)+8k+18h-36\beta-24\gamma-12i+12bc.$$

that is to say,

$$=n(7b+12c)-4b^2-9c^2-8b-15c+8k+18h-18\beta-12\gamma-12i+9t.$$

As a first verification of the preceding formulæ, we take the case where the surface is a complex one made up of several others. In this case the complex surface must be considered as having for a double line (in addition to whatever double lines the surfaces may have, considered separately) the aggregate of the curves of intersection of each pair of surfaces, on which every point of intersection of three surfaces will be a triple point. The effect of this double line must be to reduce the degree of the reciprocal of the complex surface to the sum of the degrees of the reciprocals of the simple surfaces.

We shall then have

$$n=\Sigma n_1;\quad b=\Sigma b_1+\Sigma n_1n_2;\quad c=\Sigma c_1;$$
$$h=\Sigma h_1+\Sigma c_1c_2;\quad \beta=\Sigma\beta_1;\quad \gamma=\Sigma\gamma_1+\Sigma c_1n_2;$$
$$t=\Sigma t_1+\Sigma n_1n_2n_3+\Sigma b_1n_2;\quad i=\Sigma i_1;$$
$$\rho=\Sigma\rho_1+\Sigma a_1n_2;\quad \sigma=\Sigma\sigma_1;$$
$$k=\Sigma k_1+\Sigma\frac{n_1n_2(n_1-1)(n_2-1)}{1\,.\,2}+\Sigma b_1b_2$$
$$+\Sigma n_1\,\Sigma n_1n_2n_3-3\Sigma n_1n_2n_3+\Sigma b_1\Sigma n_1n_2-2\Sigma b_1n_2;$$

and on substituting these values the equations (A) and (D) are satisfied identically, and the reduction in the degree of the reciprocal surface, caused by the curves of intersection of the simple surfaces, is

$$3\Sigma n_1^2n_2+6\Sigma n_1n_2n_3-2\Sigma n_1n_2;$$

but this is just the difference between

$$(\Sigma n_1)^3-2(\Sigma n_1)^2+(\Sigma n_1)\ \text{ and }\ \Sigma(n_1^3-2n_1^2+n_1).$$

II.—*Developable Surfaces.*

I come now to the application of the preceding theory to the case of developable surfaces. I use with respect to these surfaces the notation employed by Mr. Cayley ("Liouville," vol. x. p. 245; "Cambridge and Dublin Mathematical Journal," vol. v. p. 18). The degree of the developable surface is r.

It has a cuspidal line of the degree m, and an ordinary double line of the degree x. The simple line of contact (a) consists of n right lines. Each of those right lines meets the edge of regression once, and the line x in $r-4$ points (see "Cambridge and Dublin Mathematical Journal," vol. v. p. 25). The lines m and x intersect at the α points, which are the points of contact of stationary planes of the system; for, since there three consecutive lines of the system lie in the same plane, the intersection of the first and third of these must belong to the line x, which is the locus of the points on two consecutive lines of the system.* The tangent cone to the developable breaks up into n planes; it has, therefore, no cuspidal lines, and $\frac{n(n-1)}{1.2}$ double lines.

We have then the following table. The letters on the left-hand side of the equation refer to the notation of the preceding theory; the letters on the right-hand to the notation in the papers on developable surfaces just referred to:

$$n=r,\ a=n,\ b=x,\ c=m,\ \rho=n(r-4),\ \sigma=n,\ \kappa=0,\ \beta=\beta,\ h=h,\ i=\alpha,$$

and the quantities t, γ, k remain to be determined by the present theory. On substituting these values in the equation (A) and (D), we obtain the following system of equations :—

$$\begin{aligned}
n\,(r-2)&=n\{2+(r-4)\},\\
x\,(r-2)&=n\,(r-4)+2\beta+3\gamma+3t,\\
m\,(r-2)&=2n+4\beta+\gamma,\\
n\,(r-2)\,(r-3)&=n\{(n-1)+2x+3m-4\,(r-4)-9\},\\
x\,(r-2)\,(r-3)&=4k+nx+3mx-9\beta-6\gamma-3\alpha-2n\,(r-4),\\
m\,(r-2)\,(r-3)&=6h+mn+2mx-6\beta-4\gamma-2\alpha-3n.
\end{aligned}\qquad\text{(E)}$$

The first of these equations is verified immediately, and the fourth by the help of the equation given by Mr. Cayley ("Cambridge and Dublin Mathematical Journal," vol. v. p. 21),

$$2x+3m+n=r\,(r-1).$$

We may determine γ either from the third or from the sixth equation. That

* It was the consideration of this case which led me to include in the preceding theory the points i, of which I have never happened to meet with any other instance.

the results derived from both are identical, appears on eliminating γ between these equations, when we have

$$m(r^2 - r) = 6h + mn + 2mx + 10\beta - 2a + 5n + 2m,$$

an equation which that just given reduces to

$$6h - 3m^2 + 10\beta - 2a + 5n + 2m = 0.$$

And this equation is immediately verified by adding the two given in Mr. CAYLEY's memoir just cited,

$$6h + 8\beta + n = 3m(m-2), \qquad 2(\beta - a) = 4(m-n).$$

It appears then, that of the six equations (E), three may be verified by the theory of developable surfaces already known, while the remaining three determine the three quantities, t, the number of "points on three lines" of the system, γ, the number of points of the system through which pass another non-consecutive line of the system, and k the number of apparent double points on the nodal line of the developable. It is obvious that the corresponding reciprocal singularities may be determined in like manner, that is to say, the number of "planes through three lines," &c.

It is possible to verify the value just found for γ by investigating this quantity directly* in the case where the edge of regression of the developable is the intersection of two surfaces U and V, the former being supposed to be of the degree k, and the latter of the degree l. The points where the line joining two given points meets each of the surfaces is determined, by the method already given, from the equations,

$$\lambda^k U + \lambda^{k-1}\mu\,\Delta U + \frac{\lambda^{k-2}\mu^2}{1\,.\,2}\Delta^2 U + \&c. = 0,$$

$$\lambda^l V + \lambda^{l-1}\mu\,\Delta V + \frac{\lambda^{l-2}\mu^2}{1\,.\,2}\Delta^2 V + \&c. = 0\,;$$

but if the line joining the two given points be a line of the system, and $xyzw$ its point of contact, we have $U=0$, $V=0$, $\Delta U=0$, $\Delta V=0$. Introducing these values, and eliminating λ, μ, between the equations, we shall have the condition

* The method of investigation employed is the same as that by which Mr. CAYLEY has determined the number of points of inflexion and double tangents of plane curves, of which I have elsewhere given an account.—("Higher Plane Curves," pp. 77, 86.)

that the given line should meet the curve UV again. This condition will be of the degree $(k-2)(l-2)$ in $xyzw$, and of the degree $lk-4$ in $x'y'z'w'$. But since this condition must be satisfied for every point of the given line, if we eliminate $x'y'z'w'$ between it, the two equations of the right line, $\Delta U=0$, $\Delta V=0$, and the equation of an arbitrary plane, $\alpha x+\beta y+\gamma z+\delta w=0$, we shall have a result which will be of the form

$$\Pi(\alpha x+\beta y+\gamma z+\delta w)^{lk-4}.$$

Π will then be of the degree

$$(k-2)(l-2)+(lk-4)(k+l-3),$$

and the intersections of the surface represented by $\Pi=0$ with U, V will give the points γ. If then we write $kl=q$, $k+l=p$, the number of points γ will be

$$q(pq-2q-6p+16).$$

But I have shown ("Cambridge and Dublin Mathematical Journal," vol. v. p. 32) that in the same case

$$m=q,\ n=3q(p-3),\ r=q(p-2),\ \beta=0,$$

by the help of which values the equation

$$m(r-2)=2n+4\beta+\gamma$$

is satisfied identically. The case of developable surfaces, thus examined, proves, I think, that the numerical coefficients in at least the first and third of equations (A) and (D) have been rightly determined. It is scarcely necessary to observe, that the singularities here noticed may be sometimes replaced by others of a higher order. For example, the developable, whose edge of regression is the line of intersection of two surfaces of the second degree, has no "point on three lines," but has, instead, four "points on four lines."

As a further illustration of the theory of developable surfaces, I take the case of the developable which is the envelope of the variable plane

$$At^{\mu}+\mu Bt^{\mu-1}+\frac{\mu(\mu-1)}{1.2}Ct^{\mu-2}+\&c.=0,$$

where t is a variable parameter.

This surface has been elsewhere discussed ("Cambridge and Dublin Mathematical Journal," vol. iii. p. 169; vol. v. pp. 46, 152). Its characteristics I have there stated to be—

$$m = 3(\mu-2),\ n=\mu,\ r=2(\mu-1),\ \alpha=0,\ \beta=4(\mu-3),\ g=\tfrac{1}{2}(\mu-1)(\mu-2),$$
$$h=\tfrac{1}{2}(9\mu^2-53\mu+80),\ x=2(\mu-2)(\mu-3),\ y=2(\mu-1)(\mu-3).$$

On substituting, then, these values in equations (E), we find

$$\gamma = 6(\mu-3)(\mu-4)\ ;\ 3t=4(\mu-3)(\mu-4)(\mu-5)\ ;$$
$$k=(\mu-3)(2\mu^3-18\mu^2+57\mu-65).$$

These values may be considered as so far verifying the preceding equations, insomuch as it is evident, *à priori*, that points γ cannot exist when μ is less than 5, nor points t when μ is less than 6. I may add, that I have calculated independently the order of the conditions that the equations

$$At^{\mu-1}+(\mu-1)\,Bt^{\mu-2}+\&c.=0, \qquad Bt^{\mu-1}+(\mu-1)\,Ct^{\mu-2}+\&c.=0,$$

should have three common factors, and found the result

$$\frac{(2\mu-4)\ (2\mu-5)\ (2\mu-6)}{1\,.\,2\,.\,3}.$$

But this is exactly the sum of the numbers β, γ, t. I have similarly examined, by an independent method, the number of apparent double points in the curve represented by the conditions that the two equations just written should have two common roots, and found the result

$$\frac{(\mu-2)\ (2\mu-3)\ (2\mu-5)\ (3\mu-7)}{1\,.\,2\,.\,3}.$$

Now, since these conditions represent as well the cuspidal curve as the nodal curve x of the developable, the number of apparent double points in the complex curve should be $h+k+(mx-3\beta-2\gamma-\alpha)$; the latter number being that already found as representing the number of apparent common points of the cuspidal and nodal curve. On proceeding, however, to substitute the values already found for h, k, &c., we find

$$\frac{(\mu-2)\ (2\mu-3)\ (2\mu-5)\ (3\mu-7)}{1\,.\,2\,.\,3}=h+k+(mx-3\beta-2\gamma-\alpha)+\beta+\gamma+t,$$

instead of being

$$=h+k+(mx-3\beta-2\gamma-\alpha).$$

It is, I think, plain that we are not to attempt to reconcile these equations by supposing the value here given for k to be erroneous; but rather by considering that the method of verification employed by me gave me the actual, as well as the apparent, double points of the complex curve.

The values of γ', t', k' for the reciprocal system are found in like manner—

$$\gamma' = 2(\mu-2)(\mu-3); \ 3t' = 4(\mu-2)(\mu-3)(\mu-4);$$
$$2k' = (\mu-3)(4\mu^3 - 31\mu^2 + 77\mu - 62).$$

III.—*Singularities of the Reciprocal Surface.*

We proceed next to determine the values of the quantities ρ, σ, &c., for the reciprocal of a given surface; to verify that these values, substituted in the equations (A) and (D), will satisfy them, and thus to show that the reciprocal of the reciprocal will reduce to the degree of the given surface. We shall endeavour to determine directly as many of these singularities as we can, but we are obliged to limit ourselves to the case where the original surface has no multiple points or lines. We have seen that in this case the tangent cone drawn to the original surface from an arbitrary point is of the degree $n(n-1)$, having $n(n-1)(n-2)$ cuspidal lines, and $\dfrac{n(n-1)(n-2)(n-3)}{1.2}$ double lines. The reciprocal of this will be the section of the reciprocal surface by an arbitrary plane. Its degree will be

$$n^2 = n(n-1)\{n(n-1)-1\} - 3n(n-1)(n-2) - n(n-1)(n-2)(n-3)$$
$$= n(n-1)^2.$$

The number of cusps in the section of the reciprocal surface, found by the ordinary rule is

$$3n(n-1)\{n(n-1)-2\} - 8n(n-1)(n-2) - 3n(n-1)(n-2)(n-3)$$
$$= 4n(n-1)(n-2).$$

Since, then, any section of the reciprocal surface has this number of cusps, we learn that the reciprocal surface has a cuspidal line whose degree is

$$c' = 4n(n-1)(n-2).$$

In like manner twice the number of double points in a section of the reciprocal surface is

$$n(n-1)^2\{n(n-1)^2-1\}-n(n-1)-12n(n-1)(n-2)$$
$$=n(n-1)(n-2)(n^3-n^2+n-12).$$

Hence, then, by the same reasoning, the reciprocal surface has a double line whose degree is

$$2b'=n(n-1)(n-2)(n^3-n^2+n-12).$$

The importance of these results justifies us in giving another and more direct investigation of them. To every double or stationary point on the reciprocal surface corresponds a double or stationary tangent plane on the original surface. Let us then investigate directly the conditions fulfilled by the point of contact of a double tangent plane to a given surface. We investigate these by the same method by which we investigated the condition that a line should touch a surface. If the co-ordinates of three points be $x_1y_1z_1w_1$, $x_2y_2z_2w_2$, $xyzw$, then those of any point on the plane through the three points will be $\lambda x+\mu x_1+\nu x_2$, $\lambda y+\mu y_1+\nu y_2$, $\lambda z+\mu z_1+\nu z_2$, $\lambda w+\mu w_1+\nu w_2$; and if we substitute these values for $xyzw$ in the equation of the surface, we shall have the relation which must be satisfied for every point where this plane meets the surface. Let the result of this substitution be $[U]=0$; it may be written—

$$\lambda^n U+\lambda^{n-1}\mu\Delta_1 U+\lambda^{n-1}\nu\Delta_2 U+\frac{\lambda^{n-2}}{1.2}(\mu\Delta_1+\nu\Delta_2)^2U+\&c.,$$

where

$$\Delta_1=\left(x_1\frac{d}{dx}+y_1\frac{d}{dy}+z_1\frac{d}{dz}+w_1\frac{d}{dw}\right),\qquad \Delta_2=x_2\frac{d}{dx}+y_2\frac{d}{dy}+z_2\frac{d}{dz}+w_2\frac{d}{dw}.$$

Now since the tangent plane to a surface always meets it in a section having a double point, the condition that the plane joining the three given points should touch the surface, is found by eliminating λ, μ, ν between

$$\frac{d[U]}{d\lambda}=0,\quad \frac{d[U]}{d\mu}=0,\quad \frac{d[U]}{d\nu}=0;$$

or, in other words, the *discriminant* of the equation $[U]$. If we suppose two of the points fixed, and consider the third to be variable, then the condition so found will be the equation of the tangent planes to the surface, which can be

drawn through the line joining the two fixed points. We shall suppose the point $xyzw$ to be on the surface, and the point $x_1y_1z_1w_1$ to be taken anywhere on the tangent plane at that point; then we shall have $U=0$, $\Delta_1 U = 0$, and the discriminant will become divisible by the square of $\Delta_2 U$. For plainly, of the tangent planes, which can be drawn to a surface through any tangent line to that surface, two will coincide with the tangent plane at the point of contact of that line. If the tangent plane at $xyzw$ be a double tangent plane, then the discriminant will be divisible by the *cube* of $\Delta_2 U$. If we write, for brevity, $\Delta_1{}^2 U = A$, $\Delta_1\Delta_2 U = B$, $\Delta_2{}^2 U = C$, so as to make the coefficient of λ^{n-2} in $[U]$ to be written $A\mu^2 + 2B\mu\nu + C\nu^2$, then I say that the coefficient of the square of $\Delta_2 U$ in the discriminant of $[U]$ is

$$A(B^2 - AC)^2 \,\square,$$

where $\square$ is the discriminant of the equation when $U=0$, $\Delta_1 U=0$, $\Delta_2 U = 0$. I have verified this in the case of the equation of the third degree, and I feel that I am safe in asserting it to be true in general. In order, then, that the discriminant should be divisible by the cube of $\Delta_2 U$, some one of these factors must either vanish or be divisible by $\Delta_2 U$.

First, then, let $A = 0$, or $\Delta_1{}^2 U = 0$. This will be the case if the point $x'y'z'w'$ be taken on either of the lines which can be drawn through $xyzw$ so as to meet the surface in three consecutive points. We shall suppose, however, that the point $x'y'z'w'$ has not been so assumed, and then, as A does not contain $x_2y_2z_2w_2$, this factor may be set aside as irrelevant to the present discussion.

Secondly, let $B^2 - AC$ be divisible by $\Delta_2 U$. Let it be required to find the condition to be satisfied by the point $xyzw$ in order that this should be the case. Now if $B^2 - AC$ contain $\Delta_2 U$ as a factor, any arbitrary right line which meets the plane* $\Delta_2 U$, will meet $B^2 - AC$; and, therefore, if we eliminate $x_2y_2z_2w_2$ between these two equations and those of an arbitrary right line—

$$ax + by + cz + dw = 0, \qquad a'x + b'y + c'z + d'w = 0,$$

the result of elimination, $R = 0$, must be satisfied identically. This resultant will be of the second degree in $abcd$ and in $a'b'c'd'$; of the second, in $x'y'z'w'$, and of the $4n - 6^{th}$ in $xyzw$.

* N. B.—$x_2y_2z_2w_2$ is here considered as variable; $xyzw$, as fixed.

Now since the discriminant of $[U]$ in general represents a number of planes passing through the line joining the points $xyzw$, $x'y'z'w'$; this line will be a multiple line in the locus represented by that equation. And in the present case, where $xyxw$ satisfies the equation of the surface, and $x'y'z'w'$ that of the tangent plane at the point, it is easy to see that this line is a double line on $B^2 - AC$, and a multiple line of the degree $n(n-1)^2 - 6$ on $\square$.

If, then, the arbitrary line had been so assumed as to meet the line joining $xyzw$, $x'y'z'w'$, the condition $R = 0$ would be satisfied even if $\Delta_2 U$ were not a factor in $B^2 - AC$. The condition that the two lines should meet ($M = 0$) will be of the first degree in $abcd$, $a'b'c'd'$; $xyzw$, $x'y'z'w'$; and it is plain that R must be of the form $M^2 H = 0$. H remains a function of $xyzw$ only, and is of the $4(n-2)$ degree.

At all points then of the intersection of the surfaces $U = 0$, $H = 0$, the tangent plane must be considered as double. H is no other than the Hessian of the surface, and its intersection with U is the well-known parabolic curve, at every point of which, I have elsewhere shown ("Cambridge and Dublin Mathematical Journal," vol. iii. p. 44), the tangent plane touches the surface in two consecutive points.

We investigate in precisely the same way the condition that $\Delta_2 U$ should be a factor in $\square$. Eliminating between these two equations and those for an arbitrary line, we obtain a condition of the degree $n^3 - 2n^2 + n - 6$ in $abcd$, in $a'b'c'd'$, in $x'y'z'w'$, and of the degree $n^4 - 2n^3 + n^2 - 13n + 18$ in $xyzw$. But, as before, this condition must be of the form $M^{n^3-2n^2+n-6} J = 0$. J, then, is a function of $xyzw$ only, and is of the $(n-2)$ $(n^3 - n^2 + n - 12)$ degree. We learn hence that all the points of a surface whose tangent planes touch it also at a second distinct point, lie on the intersection of the surface U with the surface $J = 0$, which is of the $(n-2)$ $(n^3 - n^2 + n - 12)$ degree.

It is easy now to deduce hence the degree of the cuspidal and nodal curves on the reciprocal surface. To every point on the cuspidal curve will correspond a double tangent plane touching the original surface somewhere on the curve UH; and to every point on the nodal curve will correspond a plane touching somewhere on the curve UJ. The points where an arbitrary plane meets the multiple curves on the reciprocal surface correspond to the planes which can be drawn through an arbitrary point, whose points of contact lie on H or J. And

since the curve of contact of planes passing through a fixed point is of the $n(n-1)^{st}$ degree, the number of points in which this curve meets H and J will be $4n(n-1)(n-2)$ and $n(n-1)(n-2)(n^3-n^2+n-12)$; a result perfectly agreeing with that which we otherwise obtained in the beginning of this section.

We next proceed to determine ρ' and σ', the number of points in which the line of simple contact to the reciprocal surface meets its double and cuspidal curves. This is obviously equal to the number of double or stationary tangent planes which touch the given surface along an arbitrary plane section, and is therefore equal to the number of points where an arbitrary plane meets UH and UJ. Hence we have

$$\rho' = n(n-2)(n^3-n^2+n-12), \qquad \sigma' = 4n(n-2).$$

The number of points $(b'c')$ on the reciprocal surface plainly is equal to the number of points of intersection of the surfaces U, H, J; hence

$$(b'c') = 4n(n-2)^2(n^3-n^2+n-12).$$

Now of these points $(b'c')$ a certain number will be stationary points β' on the curve c'. These correspond to the case where the same tangent plane touches the surface along two consecutive points of the parabolic curve. But I have proved already (see "Cambridge and Dublin Mathematical Journal," vol. iii. p. 44) that this will happen when at such a point a line can be drawn to meet the surface in four consecutive points; and also (see "Cambridge and Dublin Mathematical Journal," vol. iv. p. 260), that all such points lie on a surface S of the degree $11n-24$. The curve US *touches* the parabolic curve UH. Hence the number of points in which U, S, H, intersect gives

$$\beta' = 2n(n-2)(11n-24).$$

Every other point $(b'c')$ will be a point γ', that is to say, a stationary point on the curve b'. For such a point corresponds to a plane which touches the original surface at one point on the parabolic curve, at another on UJ. But from the mere fact of the plane's touching at a point of the parabolic curve, it is a double tangent plane: it must then, in two ways, belong to the system which touches along the curve UJ; or, in other words, it must be a stationary plane of that system. Hence,

$$\gamma' = (b'c') - 2\beta' = 4n(n-2)(n-3)(n^3+3n-16).$$

It is also possible to determine *à priori* the number of apparent double points h' belonging to the curve c'. For we can determine the *rank* of that system; or, in other words, the degree of the reciprocal developable. Two consecutive planes which touch along the parabolic curve intersect in the line which meets the surface along three consecutive points.

Now suppose it were required to determine the degree of the surface generated by the lines which can be drawn to meet U in three consecutive points along any curve UV, where V is of the p^{th} degree; this is done by eliminating between

$$U=0, \quad V=0, \quad \Delta_1 U=0, \quad \Delta_1{}^2 U=0,$$

and the result is of the degree $np(3n-4)$.

But in the present case $p=4(n-2)$; and since the two lines which meet in three consecutive points coincide along every point of UH, this result must be a perfect square. Hence

$$r' = 2n(n-2)\ (3n-4),$$

and

$$2h' = c'^2 - c' - r' - 3\beta' = n(n-2)\ \{16n^4 - 64n^3 + 80n^2 - 108n + 156\}.$$

These are the only singularities of the reciprocal surface which I have been able to determine *à priori*, except the number of cusps and double lines on the tangent curve proper to the reciprocal surface; these follow immediately from the consideration that this line is the reciprocal to a plane section of the original surface, supposed to be of the n^{th} degree, and having no multiple points. Hence,

$$\kappa' = 3n(n-2); \quad 2\delta' = n(n-2)\ (n^2-9).$$

For the sake of convenience, we assemble into a table the results already obtained—

$$n' = n(n-1)^2.$$

$$a' = n(n-1); \quad 2b' = n(n-1)(n-2)(n^3-n^2+n-12); \quad c' = 4n(n-1)(n-2);$$

$$\rho' = n(n-2)\ (n^3-n^2+n-12); \quad \sigma' = 4n(n-2);$$

$$\kappa' = 3n(n-2); \quad 2\delta' = n(n-2)\ (n^2-9);$$

$$i'=0, \quad \beta' = 2n(n-2)\ (11n-24), \quad \gamma' = 4n(n-2)(n-3)(n^3+3n-16);$$

$$2h' = n(n-2)\ (16n^4 - 64n^3 + 80n^2 - 108n + 156).$$

Substituting these values in the equations (A) and (D), we obtain the following system of equations, remembering that $n'-2=(n-2)(n^2+1)$, we have

$$n(n-1)(n-2)(n^2+1)=3n(n-2)+n(n-2)(n^3-n^2+n-12)$$
$$+8n(n-2),$$

$$n(n-1)(n-2)\frac{n^3-n^2+n-12}{2}(n-2)(n^2+1)=n(n-2)(n^3-n^2+n-12)$$
$$+4n(n-2)(11n-24)+12n(n-2)(n-3)(n^3+3n-16)+3t',$$

$$4n(n-1)(n-2)^2(n^2+1)=8n(n-2)+8n(n-2)(11n-24)$$
$$+4n(n-2)(n-3)(n^3+3n-16),$$

$$n(n-1)(n-2)(n^2+1)(n^3-2n^2+n-3)=n(n-2)(n^2-9)$$
$$+n^2(n-1)^2(n-2)(n^3-n^2+n-12)+12n^2(n-1)^2(n-2)$$
$$-4n(n-2)(n^3-n^2+n-12)-36n(n-2),$$

$$n(n-1)(n-2)\frac{n^3-n^2+n-12}{2}(n-2)(n^2+1)(n^3-2n^2+n-3)$$
$$=4k'+\tfrac{1}{2}n^2(n-1)^2(n-2)(n^3-n^2+n-12)$$
$$+6n^2(n-1)^2(n-2)^2(n^3-n^2+n-12)-18n(n-2)(11n-24)$$
$$-24n(n-2)(n-3)(n^3+3n-16)$$
$$-2n(n-2)(n^3-n^2+n-12),$$

$$4n(n-1)(n-2)^2(n^2+1)(n^3-2n^2+n-3)$$
$$=3n(n-2)(16n^4-64n^3+80n^2-108n+156),$$
$$+4n^2(n-1)^2(n-2)+4n^2(n-1)^2(n-2)^2(n^3-n^2+n-12)$$
$$-12n(n-2)(11n-24),$$
$$-16n(n-2)(n-3)(n^3+3n-16)-12n(n-2).$$

On examining these equations it will be found that four of them are satisfied identically, while the remaining two give the values,

$$6t'=n(n-2)(n^7-4n^6+7n^5-45n^4+114n^3-111n^2+548n-960)$$
$$8k'=n(n-2)(n^{10}-6n^9+16n^8-54n^7+164n^6-288n^5+547n^4-1058n^3$$
$$+1068n^2-1214n+1464).$$

It would be desirable to test these results by obtaining the number of triple tangent planes to a surface of the n^{th} degree by a different process. I have endeavoured to determine this number by the same method by which we deter-

mined the nature of the curve of contact of double tangent planes to the surface. By this method it would be necessary to examine when the coefficient of the cube of $\Delta_2 U$ in the discriminant of $[U]$ (which is of the form $E(B^2 - AC) + F\square$), vanishes, or becomes divisible by $\Delta_2 U$. I have not succeeded, however, in deriving the theory of triple tangent planes in this way.

POSTSCRIPT.—(Added Jan. 5, 1857.)

An interesting application of the preceding theory may be made to the class of ruled surfaces, which is obtained by eliminating t between the equations

$$At^a + Bt^{a-1} + Ct^{a-2} \text{ \&c.} = 0, \qquad A't^b + B't^{b-1} + C't^{b-2} + \text{\&c.} = 0,$$

where A, B, &c., are functions of the co-ordinates of the first degree.

Mr. CAYLEY has proved ("Cambridge and Dublin Mathematical Journal," vol. vii. p. 171) that the reciprocal of every ruled surface is a surface of the same degree. In fact, since every tangent plane contains a generating right line, the number of tangent planes which can be drawn through an arbitrary right line is the same as the number of generating right lines which meet the arbitrary right line. Now if $a + b = \mu$, the degree of the surface we are now studying is μ, and it is proved by the methods which I have employed ("Quar. Jour. of Mathematics," vol. i. p. 252) that the surface contains a double line of the degree $\frac{(\mu-1)(\mu-2)}{1.2}$, on which there are $\frac{(\mu-2)(\mu-3)(\mu-4)}{1.2.3}$ triple points. The number of apparent double points of this line investigated by the same methods came out $\frac{(\mu-1)(\mu-2)(\mu-3)(3\mu-8)}{1.2.3.4}$, but I have reason to believe (see p. 471) that this number includes the triple points; wherefore, subtracting their number, as previously determined, we have remaining $\frac{(\mu-2)(\mu-3)(\mu^2-5\mu+8)}{2.4}$ for the true number of apparent double points. And it will be found that these values agree with the two following equations, derived from equations A and D (p. 465),

$$(b-a)(n-2) = 3t - \kappa, \quad (2b-a)(n-2)(n-3) = 8k - 2\delta;$$

for we have

$$a=2(\mu-1),\quad b=\frac{(\mu-1)\,(\mu-2)}{1\,.\,2},\quad \kappa=3(\mu-2),\quad \delta=2(\mu-2)\,(\mu-3).$$

IV.—*Theory of Higher Multiple Lines.*

In order to complete the subject, I give another independent method of investigating the theory of reciprocal surfaces, which is that which I first employed ("Cambridge and Dublin Mathematical Journal," vol. ii. p. 66). The degree of the reciprocal surface, being measured by the number of points in which an arbitrary line meets that surface, is equal to the number of tangent planes which can be drawn through an arbitrary line to the original surface. Now the points of contact of such planes lie on the polar surface of any point on the arbitrary right line. Take then the polar surfaces of any two points on the arbitrary line; then the intersection of these two surfaces of the $(n-1)^{st}$ degree with the given surface determines $n\,(n-1)^2$ points. The degree of the reciprocal of a surface of the n^{th} degree is therefore $n\,(n-1)^2$. Should the surface have a double point, this being an ordinary point on each of the two polar surfaces, will count for two intersections of the three surfaces. A double point, therefore, diminishes by two the degree of the reciprocal of a surface.

Ex. 1. The surface of the third degree,

$$\left(\frac{x}{a}\right)^{-1}+\left(\frac{y}{b}\right)^{-1}+\left(\frac{z}{c}\right)^{-1}+\left(\frac{w}{d}\right)^{-1}=0,$$

has four double points, namely, the four points where three of the planes x, y, z, w intersect: and the reciprocal, whose equation is of the form

$$\left(\frac{x}{\alpha}\right)^{\frac{1}{2}}+\left(\frac{y}{\beta}\right)^{\frac{1}{2}}+\left(\frac{z}{\gamma}\right)^{\frac{1}{2}}+\left(\frac{w}{\delta}\right)^{\frac{1}{2}}=0,$$

is reduced by the four double points to the fourth degree from the twelfth, which it otherwise would have been.

Ex. 2. A surface of the fourth degree (for example, Fresnel's wave surface) may have sixteen double points, and in this case the degree of its reciprocal will be reduced from the thirty-sixth to the fourth.

If the tangent cone at the double point break up into two planes, then, such a double point diminishes the degree of the reciprocal by three ; since the two polar surfaces both touch the line of intersection of the two planes, which passes through three consecutive points of the given surface. Should the two planes coincide, the degree of the reciprocal will be diminished by six.

It remains to trace by this method the effect of a double or other multiple line in depressing the degree of the reciprocal. In this case each of the polar surfaces will pass through the line in question, and we are led to the problem, "Three surfaces have common a certain line,—in how many other points do they intersect?" It will be convenient to commence with the case when the multiple line is a right line.

Before we discuss this problem, however, it is useful to examine carefully the nature of the intersection of the curve of simple contact with the double line. If a surface have a double line, the tangent cone to it from any point consists of the plane containing the point and the double line (reckoned twice), and of the cone of simple contact whose degree is $n^2 - n - 2$. If now we consider the intersection of this latter cone by the plane in question, it is evident that $(n-2)(n-3)$ of the lines of intersection are the tangents from the point to the curve (of degree $n-2$), in which the plane cuts the given surface ; and before investigation it was natural to think that the remaining $4(n-2)$ lines must be the lines (reckoned four times) to the $(n-2)$ points, where the double line meets this curve. Let, however, the equation of a surface containing a double line be

$$Ax^2 + Bxy + Cy^2 + Dy^3 + \&c. = 0$$

(where A, B, C are functions of the co-ordinates of the degree $n-2$), then the discriminant of this equation, with regard to y, may represent any tangent cone to the surface, since the plane y is arbitrary. This discriminant will contain x^2 as a factor, and if we divide by x^2, and then make $x = 0$, the remainder will be $(B^2 - 4AC)\,C^2\phi$, where ϕ is the discriminant of the equation

$$C + Dy + \&c. = 0.$$

This proves that the section of the simple tangent cone by the plane a consists of the lines which touch the plane section, of the lines (reckoned twice) to the points where this section is met by the double line, and besides of lines to what

I have called the cuspidal points on the double line (see "Cambridge and Dublin Mathematical Journal," vol. ii. p. 72), viz., the points at which the two tangent planes to the surface coincide; for these points are determined by the condition $B^2 = 4AC$.* It will be found, in like manner, that if the surface have a triple right line, there will be on that right line $4(n-3)$ points, at which two tangent planes coincide, and that the lines to these points are edges of the cone of simple contact; and, generally, that if the surface have a right line of the degree p of multiplicity, there will be on that right line $2(p-1)(n-p)$ points, the lines to which are edges of the cone of simple contact.

We return now to the case of a surface having a double line. Any two polar surfaces will then pass through that line, and the question is, in how many points not on that line will they intersect the original surface.

We give first the solution of the question. Three surfaces, whose degrees are a, b, c, have a right line common,—in how many other points do they intersect? The intersection of the first two surfaces consists of that line and of a curve of the degree $ab-1$, which latter meets the third surface in $c(ab-1)$ points. But a certain number of these points will lie on the right line in question. In fact, let $Ax+By=0$, $Cx+Dy=0$, represent two surfaces having a right line in common, and of the degrees a and b respectively, then at the $a+b-2$ points, where the right line xy meets the surface $AD=BC$, the two surfaces will have the same tangent plane, and therefore (see "Cambridge and Dublin Mathematical Journal," vol. v. p. 34) this right line will meet the remaining $(ab-1)$ curve of intersection. Subtracting this number $(a+b-2)$ from the number $c(ab-1)$ previously found, we learn, that if three surfaces have a right line common, this will replace $a+b+c-2$ of the points of intersection.

Let us now apply this theory to the case with which we are concerned. The two polar surfaces intersect in the double right line, and also in a curve of degree $(n-1)^2-1$, which, according to the theory just explained, meets that right line in $2n-4$ points; namely, the cuspidal points on the double line. Since at each of these points the two polar surfaces will touch the original sur-

* It was in the manner given in the text that I was led in the year 1846 to the consideration of these cuspidal points. It is obvious that this includes a theorem concerning discriminants which has been since stated by M. JOACHIMSTAL.

face at a point in a double line, each of these points counts for three among the intersection of the three surfaces. The points then not on the double line in which the three surfaces intersect is $n\{(n-1)^2-1\}-3\times(2n-4)$. Or the double line diminishes the degree of the reciprocal by $7n-12$, as we proved otherwise, p. 467. Or again, the intersection of the given surface with one of the polar surfaces consists of the double line, and of a curve of the degree $n(n-1)-2$, meeting the right line in $3n-6$ points. For a surface $Ax^2+Bxy+Cy^2$ having a double line, meets any other $Dx+Ey$ passing through that line, in a curve which meets that line in the $a+2b-4$ points where the line meets $AE^2-BDE+CD^2$. Of the $3n-6$ points, $2n-4$ are the cuspidal points, the remaining $n-2$ are the points where the line meets a certain plane section. And the points of intersection (*not* on the double line) of the three surfaces are $(n-1)\{n(n-1)-2\}-2(2n-4)-(n-2)$, as has been already found.

In general, if a surface have a right line of the degree p of multiplicity, this will be a $(p-1)$ multiple line on each of the two polar surfaces, which will intersect besides in a curve of the degree $(n-1)^2-(p-1)^2$. And the latter curve will meet the right line in the $2(p-1)(n-p)$ points of special contact previously noticed. It will, therefore, meet the surface in points not on the multiple line $n\{(n-1)^2-(p-1)^2\}-(p-1)\,2(p-1)\,(n-p)$. The multiple line, therefore, diminishes the degree of the reciprocal by $(3p+1)(p-1)n-2p(p^2-1)$. Or again, the original surface intersects any polar surface in a curve of the degree $n(n-1)-p(p-1)$ meeting the right line in the $2(p-1)(n-p)$ points of special contact, and in $n-p$ other points. And the three surfaces will intersect in points not on the multiple line,

$$(n-1)\{n(n-1)-p(p-1)\}-p\,.\,2(p-1)\,(n-p)-(p-1)\,(n-p),$$

which agrees with the result obtained already.

I next investigate the diminution produced by the multiple line in the number of cuspidal and double edges of the cone of simple contact.

The cuspidal edges answer, as has been before proved, to the intersections of the surface with a first and second polar surface; and the multiple line is of the degree $p-2$ on the latter surface. I have satisfied myself that the formula for the number of intersections of the three surfaces is,

$$(n-2)\{n(n-1)-p(p-1)\}-(p-2)\,2(p-1)\,(n-p)-(p-1)\,(n-p).$$

The diminution, therefore, in the number of cuspidal edges of the cone is,

$$3(p-1)^2 n - p(p-1)\ (2p-1).$$

To investigate the diminution of the number of double points: we have already seen that the cone of simple contact intersects the plane through the multiple line in three distinct classes of edges, viz.: a, tangents to the plane section; b, lines to the points where the multiple line meets that section; and c, lines to the points of special contact. Now, I have satisfied myself that the formula for the intersections of the curve of contact with the curve which determines the point of contact of double edges of the tangent cone is

$$(n-2)\ (n-3)\{n(n-1)-p(p-1)\}-p(p-1)a-2(p-1)\ (p-2)b \\ -(p-2)\ (p-3)\,c.$$

Putting in this formula the values

$$a=(n-p)\ (n-p-1), \qquad b=n-p, \qquad c=2(p-1)\ (n-p),$$

we obtain for twice the reduction in the number of double edges,

$$2p(p-1)n^2-(p-1)\ (14p-8)n-p(p-1)\ (p^2-9p+2).$$

Now since the degree of the tangent cone is reduced from $n(n-1)$ to $n(n-1)-p(p-1)$, the degree of its reciprocal is reduced for this reason alone, by

$$2p(p-1)n^2-2p(p-1)n-p(p-1)\ (p^2-p+1).$$

Subtract from this twice the reduction in the number of double edges, and three times the reduction in the number of cuspidal edges, and we get the same value as before for the reduction in the degree of the reciprocal.

I now proceed to examine the effect on the degree of the reciprocal produced by a multiple curve in general, and commence with the case of a double curve, which is supposed to be of the degree μ, and rank ρ (the rank being the degree of the developable generated by its tangents). The intersection of the two polar surfaces consists of the curve μ, and of another curve of the degree $(n-1)^2-\mu$, and the question is, in how many points do these two curves intersect. Now, I say, in general that if two surfaces of degrees m and n have the curve μ common, it will intersect the remaining curve of intersection in $\mu(m+n-2)-\rho$ points. In fact, we might seek the points on the curve μ where the surfaces touch, by first finding the locus of points such that the intersection of its polar

planes, with respect to the two surfaces, shall meet an arbitrary right line, and this is immediately found to be a surface of the degree $(m+n-2)$. Now the curve μ meets this surface either in points in which the two given surfaces have common tangent planes, or in the ρ points, the tangent to μ at which meets the arbitrary right line.* Hence, in the case we are discussing, the curve μ intersects the curve $(n-1)^2-\mu$ in $\mu(2n-4)-\rho$ points. But of these ρ are the points the tangents to μ at which meet the arbitrary line through which we are seeking how many tangent planes can be drawn to the surface, and the remaining $\mu(2n-4)-2\rho$ are cuspidal points. And the formula for the intersection of the three surfaces is

$$n\{(n-1)^2-\mu\}-2\rho-3\{\mu(2n-4)-2\rho\},$$

or the diminution in the degree of the reciprocal is $\mu(7n-12)-4\rho$.

The surface is intersected by any polar surface in the curve μ (reckoned twice), and in a curve $n(n-1)-2\mu$, which meets μ in the cuspidal points, and in the $\mu(n-2)$ other points, where the curve meets the second polar surface.

In like manner, if the curve μ be of the order p of multiplicity on the given surface, the points c of special contact will be in number

$$2\mu(p-1)(n-p)-p(p-1)\rho;$$

the points b will still be $\mu(n-p)$, and the edges a, where the cone of simple contact intersects the cone standing on the multiple line, will be

$$\mu(n-p)(n-p-1)-\mu(\mu-1)p(p-1)+p(p-1)\rho.$$

Hence, proceeding precisely as before, we obtain for the reduction in the degree of the reciprocal,

$$\mu(p-1)(3p+1)n-2\mu p(p^2-1)-p^2(p-1)\rho;$$

for the reduction in the number of cuspidal edges of the cone of simple contact,

$$\mu\{3(p-1)^2n-p(p-1)(2p-1)\}-p(p-1)(p-2)\rho;$$

* I owe to Mr. CAYLEY this demonstration of a theorem, of which I have given a less satisfactory proof ("Camb. and Dub. Math. Journal," vol. v. p. 35).

and for twice the reduction in the number of its double edges,

$$2\mu p(p-1)n^2-\mu(p-1)(14p-8)n+\mu p(p-1)(8p-2)-p^2(p-1)^2\mu^2 \\ +p(p-1)(4p-6)\rho.$$

As a verification of this formula, let the surface n consist of p surfaces of the m^{th} degree, all having the same curve μ for their complete intersection, then $\mu=m^2$, $n=pm$, $\rho=2m^2(m-1)$, and the formula for the reduction in the degree of the reciprocal becomes

$$m^2\{p(p-1)(3p+1)m-2p(p^2-1)-2p^2(p-1)(m-1)\} \\ =p(p^2-1)m^3-2p(p-1)m^2.$$

But this is the difference between $mp(mp-1)^2$ and $mp(p-1)^2$.

I have endeavoured to apply this theory also to the class of ruled surfaces which I have considered ("Cambridge and Dublin Mathematical Journal," vol. viii. p. 45) generated by a right line resting on three directrices; and I have succeeded in verifying the theory in the case where two of the directrices are right lines. In this case, if the degree of the third directrix be μ, the surface is of the degree 2μ, and each of the right lines are multiples of the degree μ. Now it is easy to see that the effect of two non-intersecting multiple lines, in diminishing the degree of the reciprocal, is the sum of their separate effects, and therefore, putting $p=\mu$, $n=2\mu$, in the formula already obtained, the degree of the reciprocal is reduced by $8\mu(\mu-1)$, but this is exactly the difference between $2\mu(2\mu-1)^2$ and 2μ. It is to be observed, however, that the ruled surface in question has, as I have proved in the memoir referred to, not only the two directrices for multiple lines, but has likewise a certain number of generatrices which are double lines, and it is necessary to show that these have no effect in depressing the degree of the reciprocal. Let there be λ such lines; now it is evident that the degree of the tangent cone is less than it otherwise could have been by 2λ, while I shall show that the number of cuspidal edges of this cone is less by 6λ than it otherwise would have been. For we have proved that the number of cuspidal edges is diminished by three times the number of points where each double line meets the second polar surface, whose degree is $2\mu-2$: but since the directrices are multiple lines on that surface of the degree $\mu-2$, subtracting twice this number from $2\mu-2$, there remain but

two points on each of the double lines which affect the number of cuspidal edges. The number of stationary tangent planes continues $=0$ as before; but in any cone having no stationary tangent planes, if we diminish the degree of the cone by any number, and the number of its cuspidal edges by three times the same number, the degree of its reciprocal is not altered.

I am not able in general to apply this theory to the next simplest class of ruled surfaces, viz., those generated by a right line resting on one right line and two curvilinear generatrices of degrees μ, μ' respectively. The degree of the surface will be $2\mu\mu'$, and the right line will be multiple of the order $\mu\mu'$, and the curves of the orders μ', μ, respectively. There will be a double curve of the degree [at least?] $\frac{\mu\mu'(\mu-1)(\mu'-1)}{2}$, which each generating line meets in $(\mu-1)(\mu'-1)$ points. A certain number of generators are also double lines (see "Cambridge and Dublin Mathematical Journal," vol. viii. p. 46). I can satisfactorily explain the case where $\mu=\mu'=2$, but, as I have said, I cannot completely account for the general case.

I have also examined the ruled surface generated by a right line resting twice on a given curve of degree μ, and once on a right line. This will have the curve and right line for multiple lines, and, in addition, a double curve of the degree [at least?] $\frac{\mu(\mu-1)(\mu-2)(\mu-3)}{2\,.\,4}$. I can satisfactorily account for the case $\mu=3$, and also when the curve is the intersection of two surfaces of the second degree; but I do not know the theory of the general case.

NOTE.—*February* 12, 1857.—I have just received the "Quarterly Journal of Mathematics," No. 5, which contains a paper by Professor SCHLÄFLI, going over some of the ground traversed in the present memoir. In particular, Dr. SCHLÄFLI obtains the values given (p. 477) for r', n', b', c', β', and γ';—of these I have already published n', c' ("Cambridge and Dublin Mathematical Journal," vol. iv. p. 188), and given methods which lead to the determination of b', β', γ' ("Mathematical Journal," vol. iv. pp. 119, 260).

Dr. SCHLÄFLI does not determine the number of triple tangent planes t', but he gives the following equation,

$$4A + 6t' = n(n-2)\{n^7 - 4n^6 + 7n^5 - 45n^4 + 118n^3 - 115n^2 + 508n - 912\},$$

where A is the rank of the developable formed by the double tangent planes. Now A is given by the equation

$$A = b'^2 - b' - 2k' - 6t' - 3\gamma';$$

and on substituting for these numbers the values already given, we find

$$A = 4n(n-2)(n-3)(n^2 + 2n - 4),$$

a value which satisfies Dr. SCHLÄFLI's equation. If the arguments by which he has arrived at it turn out to be well-founded (a point which I have not yet had time to consider), there seems reason to hope that the theory of reciprocal surfaces here given will admit of considerable simplification.